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On the origin of neutrino flavour symmetry

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Abstract: We study classes of models which are based on some discrete family symmetry which is completely broken such that the observed neutrino flavour symmetry emerges indirectly as an accidental symmetry. For such "indirect" models we discuss the D-term flavon vacuum alignments which are required for such an accidental flavour symmetry consistent with tri-bimaximal lepton mixing to emerge. We identify large classes of suitable discrete family symmetries, namely the $\Delta(3n^2)$ and $\Delta(6n^2)$ groups, together with other examples such as $Z_7 \rtimes Z_3$. In such indirect models the implementation of the type I see-saw mechanism is straightforward using constrained sequential dominance. However the accidental neutrino flavour symmetry may be easily violated, for example leading to a large reactor angle, while maintaining accurately the tri-bimaximal solar and atmospheric predictions.

Keywords: Beyond Standard Model, Neutrino Physics, Discrete and Finite Symmetries

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1 Introduction

It is well known that the solar and atmospheric neutrino data are consistent with so-called tri-bimaximal (TB) mixing [1],

$$U_{TB} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \qquad (1.1)$$

The ansatz of TB lepton mixing matrix is interesting due to its symmetry properties which seem to call for a possibly discrete non-Abelian family symmetry in nature [2]. There has been a considerable amount of theoretical work in this direction [3–13]. In the neutrino flavour basis (i.e. diagonal charged lepton mass basis), it has been shown that the TB neutrino mass matrix is invariant under S, U transformations [14]

$$M^{\nu}_{TB} = SM^{\nu}_{TB}S^T = UM^{\nu}_{TB}U^T . \qquad (1.2)$$

A very straightforward argument [15] shows that this neutrino flavour symmetry group has only four elements corresponding to Klein's four-group $Z_2^S \times Z_2^U$. By contrast the diagonal charged lepton mass matrix (in this basis) satisfies a diagonal phase symmetry T. The matrices S, T, U form the generators of the group S_4 in the triplet representation, while the A_4 subgroup is generated by S, T.

Recently there have been two apparently conflicting claims in the literature concerning the nature of the underlying family symmetry G_f of the Lagrangian responsible for the TB lepton mixing matrix. It was originally claimed that S_4 is the minimal family symmetry describing leptons with TB mixing [14]. However this claim has recently been challenged in [16] where it is argued that TB lepton mixing could arise from many possible candidate family symmetries which need not contain S_4 as a subgroup. Most recently the original claim has been clarified to include a discussion of A_4 and S_3 in addition to S_4 [17].

In this paper, motivated by the above debate, we discuss the relation between the observed flavour symmetry of the neutrino mass matrix $Z_2^S \times Z_2^U$ and the underlying family symmetry of the Lagrangian G_f. We show that the flavour symmetry of the neutrino mass matrix may originate from two quite distinct classes of models. The first class of models, which we call direct models, are based on an A_4 or S_4 family symmetry, some of whose generators are directly preserved in the lepton sector and are manifested as part of the observed flavour symmetry. The second class of models, which we call indirect models, are based on any family symmetry G_f which is completely broken in the neutrino sector, while the observed neutrino flavour symmetry $Z_2^S \times Z_2^U$ in the neutrino flavour basis emerges as an accidental symmetry which is an indirect effect of the family symmetry G_f. In such indirect models the flavons responsible for the neutrino masses break G_f completely so that none of the generators of G_f survive in the observed flavour symmetry $Z_2^S \times Z_2^U$.

In the direct models, the symmetry of the neutrino mass matrix in the neutrino flavour basis (henceforth called the neutrino mass matrix for brevity) is a remnant of the S_4 symmetry of the Lagrangian. For direct models, it is correct to say [14] that the Lagrangian should contain S_4 as a subgroup, where the generators S, U are preserved in the neutrino sector, while the diagonal generator T is preserved in the charged lepton sector. For example $PSL(2,7) = \Sigma(168)$ [15] contains S_4 as a subgroup. If the family symmetry of the underlying Lagrangian is A_4, then in some cases this can lead to a direct model where the T generator of the underlying Lagrangian symmetry is preserved in the charged lepton sector, while the S generator is preserved in the neutrino sector, with the U transformation of S_4 emerging as an accidental symmetry due to the absence of flavons in the $\mathbf{1}', \mathbf{1}''$ representations of A_4 [10]. Typically direct models satisfy form dominance [11], and require flavon F-term vacuum alignment, permitting an SU(5) type unification [10].

The main focus of this paper is on the indirect models in which the underlying family symmetry of the Lagrangian G_f is completely broken, and the flavour symmetry of the neutrino mass matrix $Z_2^S \times Z_2^U$ emerges entirely as an accidental symmetry, due to the presence of flavons with particular vacuum alignments proportional to the columns of U_{TB}, where such flavons only appear quadratically in effective Majorana Lagrangian. We emphasise that such vacuum alignments can be elegantly achieved using D-term vacuum alignment, and catalogue the possible choices of discrete family symmetry G_f which are consistent with this mechanism, namely the $\Delta(3n^2)$ and $\Delta(6n^2)$ groups, together with other examples such as $Z_7 \rtimes Z_3$. Although the presence of the underlying family symmetry

G_f is crucial for producing such vacuum alignments, we shall show that the neutrino flavour symmetry $Z_2^S \times Z_2^U$ in the neutrino flavour basis does not arise as a subgroup of G_f but rather accidentally. However, since the family symmetry G_f only partly enforces the required vacuum alignments, the $Z_2^S \times Z_2^U$ flavour symmetry can be easily violated, possibly leading to a large reactor angle while accurately preserving the tri-bimaximal solar and atmospheric predictions. The see-saw mechanism can be implemented in indirect models using Constrained Sequential Dominance (CSD) [4, 18] and such models typically permit $SO(10)$ type unification [6].

In section 2 we give a novel derivation of the flavour symmetry of the neutrino mass matrix which enables the flavons of the indirect models to be readily identified. In section 3 we briefly discuss flavons of the direct models, then introduce the flavons of the indirect models. In section 4 we show how D-term vacuum alignment may be used for indirect models, and identify the possible classes of family symmetry which are consistent with a particularly simple and useful term in the flavon potential. Section 5 shows how the type I see-saw mechanism can be applied to indirect models using CSD. Section 6 concludes the paper.

2 The flavour symmetry of the neutrino mass matrix

In this section we give a novel derivation of the flavour symmetry of the neutrino mass matrix in the neutrino flavour basis which enables the flavons of indirect models to be easily identified. In the neutrino flavour basis, in which the charged lepton mass matrix is diagonal and the TB mixing arises from the neutrino sector, the effective neutrino mass matrix, denoted by M^ν_{TB}, may be diagonalised as,

$$M^\nu_{\rm diag} = U^T_{TB} M^\nu_{TB} U_{TB} = {\rm diag}\,(m_1,\ m_2,\ m_3)\,. \tag{2.1}$$

Given $U_{\rm TB}$, this enables M^ν_{TB} to be determined in terms of neutrino masses,

$$M^\nu_{TB} = m_1 \Phi_1 \Phi_1^T + m_2 \Phi_2 \Phi_2^T + m_3 \Phi_3 \Phi_3^T\ , \tag{2.2}$$

corresponding to the orthonormal column vectors

$$\Phi_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix},\quad \Phi_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\quad \Phi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tag{2.3}$$

which are just equal to the columns of U_{TB},

$$U_{TB} = (\Phi_1, \Phi_2, \Phi_3) \tag{2.4}$$

with the orthonormality relations,

$$\Phi_i^T \Phi_j = \delta_{ij}. \tag{2.5}$$

It is convenient to define the matrices

$$G_i = \Phi_i \Phi_i^T, \tag{2.6}$$

in terms of which the neutrino mass matrix is simply written,

$$M^{\nu}_{TB} = m_1 G_1 + m_2 G_2 + m_3 G_3, \tag{2.7}$$

and

$$\begin{aligned} U^T_{TB} G_1 U_{TB} &= \mathrm{diag}\,(1,0,0)\,, \\ U^T_{TB} G_2 U_{TB} &= \mathrm{diag}\,(0,1,0)\,, \\ U^T_{TB} G_3 U_{TB} &= \mathrm{diag}\,(0,0,1)\,. \end{aligned} \tag{2.8}$$

We aim to find the symmetry transformations G which leave the TB neutrino mass matrix in eq. (2.2) invariant, as in eq. (1.2), which implies that,

$$G \Phi_i \Phi_i^T G^T = \Phi_i \Phi_i^T. \tag{2.9}$$

Eq. (2.9) implies that

$$G \Phi_i = \eta_i \Phi_i, \tag{2.10}$$

where $\eta_i = \pm 1$. For a given choice of η_i we can easily find the corresponding real orthogonal matrix G, since it is diagonalised by the matrix $U_{TB} = (\Phi_1, \Phi_2, \Phi_3)$ whose columns are the eigenvectors Φ_i with eigenvalues η_i,

$$U^T_{TB} G U_{TB} = \mathrm{diag}\,(\eta_1, \eta_2, \eta_3)\,. \tag{2.11}$$

Then, by inverting eq. (2.11), we find a set of eight choices of G corresponding to the set of choices of η_i,

$$G \in G_{\eta_1,\eta_2,\eta_3} \equiv \eta_1 G_1 + \eta_2 G_2 + \eta_3 G_3. \tag{2.12}$$

For example, $G_{+++} = I$, the unit matrix, and we can define $S = G_{-+-}$, $U = G_{--+}$, which are explicitly given as,

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \qquad U = -\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.13}$$

We can write

$$S = -I + 2G_2, \quad U = -I + 2G_3. \tag{2.14}$$

In general the multiplication law for the G_{η_1,η_2,η_3} is extremely simple,

$$G_{\eta_1,\eta_2,\eta_3} G_{\eta'_1,\eta'_2,\eta'_3} = G_{\eta_1\eta'_1,\eta_2\eta'_2,\eta_3\eta'_3}. \tag{2.15}$$

Eq. (2.15) follows from

$$G_i G_j = \delta_{ij} G_i, \tag{2.16}$$

which follows from the orthonormality relations in eq. (2.5), as does,

$$G_i \Phi_j = \delta_{ij} \Phi_i. \tag{2.17}$$

From eq. (2.15) it is easy to see that S, U form a four element group together with the unit matrix I, with the fourth element $X = G_{+--}$ being given by $X = SU = US$, where all elements have determinant given by $\eta_1\eta_2\eta_3$ equal to plus one. The remaining four transformations with negative determinant given by $-I, -S, -U, -X$ clearly do not form a group. To summarise, Klein's four-group $Z_2^S \times Z_2^U$ is the flavour symmetry of the TB neutrino mass matrix with elements

$$G \in (G_{+++}, G_{-+-}, G_{--+}, G_{+--}) \equiv (I, S, U, X) \tag{2.18}$$

in the notation of eq. (2.12).

3 Flavons of direct and indirect models

3.1 The flavour symmetry problem

The typical Lagrangian (or superpotential) of interest generically consists of two parts, the Yukawa sector and the Majorana sector. The Yukawa sector is of the form,

$$\mathcal{L}^{\text{Yuk}} \sim \psi_i Y_{ij}^{\text{Yuk}} \psi_j^c H \,, \tag{3.1}$$

while the effective Majorana sector is of the form

$$\mathcal{L}^{\text{Maj}} \sim \psi_i Y_{ij}^{\text{Maj}} \psi_j H H \,, \tag{3.2}$$

where Y_{ij}^{Yuk} and Y_{ij}^{Maj} are Yukawa and Majorana couplings, respectively, while H are Higgs fields. When the Higgs develop their VEVs, and ψ are identified with left-handed lepton fields L, while ψ^c are identified with charged conjugated right-handed charged leptons such as e^c, the Yukawa operators lead to the charged lepton mass matrix $M_{ij}^e \propto Y_{ij}^e$, while the effective Majorana operators lead to a neutrino Majorana mass matrix $M_{ij}^\nu \propto Y_{ij}^\nu$.

We have already seen that the TB neutrino mass matrix is invariant under S, U transformations in eq. (2.13) while the diagonal charged lepton mass matrix is invariant under the phase transformation $T = \text{diag}(1, \omega^2, \omega)$ where $\omega = e^{2\pi i/3}$. At first sight it appears paradoxical that $\mathcal{L}^{\text{Yuk}}$ and $\mathcal{L}^{\text{Maj}}$ could respect different flavour symmetries since the lepton doublets L are common to both. Indeed, with Y_{ij}^{Yuk} and Y_{ij}^{Maj} being simply numbers, a symmetry transformation $L \to VL$ in one sector would entail an identical symmetry transformation in the other. Thus V would be a symmetry of both the neutrino and the charged lepton mass matrix.

The resolution to this problem is intrinsically related to the origin of the Yukawa couplings. If the Yukawa couplings are generated dynamically by the VEVs of flavon fields, then it is possible to have different symmetries in the Yukawa and Majorana sectors. The idea is that the complete high energy theory Lagrangian, including both $\mathcal{L}^{\text{Yuk}}$ and $\mathcal{L}^{\text{Maj}}$, would both respect some family symmetry G_f due to the presence of flavons ϕ^{Yuk} and ϕ^{Maj}, where ϕ^{Yuk} appears in $\mathcal{L}^{\text{Yuk}}$ while ϕ^{Maj} appears in $\mathcal{L}^{\text{Maj}}$,

$$\mathcal{L}^{\text{Yuk}} \sim \psi \phi^{\text{Yuk}} \psi^c H \,, \tag{3.3}$$

$$\mathcal{L}^{\text{Maj}} \sim \psi \phi^{\text{Maj}} \psi H H \,, \tag{3.4}$$

where both terms are invariant under G_f, and we have suppressed flavour indices, order unity coefficients and dimensional mass scales. Generically $\phi^{\rm Yuk}$ and $\phi^{\rm Maj}$ may represent either a single flavon or a polynomial of flavons of a particular type. When the flavons develop VEVs the family symmetry is spontaneously broken in such a way that the full family symmetry is not apparent in either of the low energy Lagrangians, only the observed flavour symmetries corresponding to S, U in $\mathcal{L}^{\rm Maj}$ and T in $\mathcal{L}^{\rm Yuk}$.

3.2 The flavons of direct models

In direct models, as discussed in the Introduction, we should have $G_f = S_4$ or a group that contains S_4 as a subgroup (or $G_f = A_4$ as discussed below). In this approach one seeks to identify the symmetry generators S, U, T respected by $\mathcal{L}^{\rm Maj}$ and $\mathcal{L}^{\rm Yuk}$, below the $G_f = S_4$ symmetry breaking scale, with the original generators contained in S_4. One introduces three types of flavon denoted as ϕ_S, ϕ_U, ϕ_T, which each preserves a particular generator, in other words the VEVs of these flavons are eigenvectors of different generators of $G_f = S_4$ with eigenvalues of $+1$,

$$S\langle\phi_S\rangle = +1\langle\phi_S\rangle, \quad U\langle\phi_U\rangle = +1\langle\phi_U\rangle, \quad T\langle\phi_T\rangle = +1\langle\phi_T\rangle, \tag{3.5}$$

where $\phi_{S,T} \sim \mathbf{3}$ are in the triplet representation and $\phi_U \sim \mathbf{2}$ are in the doublet representation of S_4. With eq. (3.5) the flavon VEVs $\langle\phi_S\rangle$ and $\langle\phi_U\rangle$ are *both* left invariant under S *as well as* under U.[1] In addition, the singlet flavon ϕ_I VEV preserves all the generators. In the effective theory the flavons $\phi_{S,U}$ enter the effective lepton operator linearly, so that $\phi^{\rm Maj}$ is a linear combination of ϕ_S, ϕ_U and ϕ_I, while the flavon ϕ_T enters the Yukawa operators linearly, with $\phi^{\rm Yuk} \sim (\phi_T + \phi_I)$,

$$\mathcal{L}^{\rm Yuk} \sim \psi(\phi_T + \phi_I)\psi^c H\,, \tag{3.6}$$

$$\mathcal{L}^{\rm Maj} \sim \psi(\phi_S + \phi_U + \phi_I)\psi HH, \tag{3.7}$$

where in the Majorana Lagrangian $\psi \sim \mathbf{3}$ represents lepton electroweak doublets in the triplet representation of S_4, the Higgs electroweak doublets $H \sim \mathbf{1}$ are S_4 singlets and we have suppressed flavour indices, order unity coefficients and dimensional mass scales. As a consequence, in direct models, all the flavon VEVs contained in $\phi^{\rm Yuk}$ preserve the T generator, while all the flavon VEVs contained in $\phi^{\rm Maj}$ preserve the S, U generators of the original family symmetry G_f. Depending on the full symmetries of the model, ϕ_I may be replaced by a pure number, leading to different suppressions between the terms in the Yukawa and/or Majorana Lagrangian.

The type I see-saw mechanism [19] in direct models exploits the flavons ϕ_S, ϕ_U, ϕ_I in eq. (3.5). In general we could consider an S_4 model of the following form, suppressing

[1] The generators of the doublet representation are $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$. Therefore the VEV of the doublet ϕ_U is trivially also an eigenvector of S with eigenvalue $+1$. Concerning the triplets in the basis of eq. (2.13) the eigenvector of S with eigenvalue $+1$ is $\langle\phi_S\rangle \propto \Phi_2$. In S_4 there are two distinct triplet representations which differ in the sign of the U generator in eq. (2.13). In the case that ϕ_S is in the triplet representation which corresponds to the positive sign of U, then $\langle\phi_S\rangle \propto \Phi_2$ also preserves the U generator.

flavour indices, order unity coefficients and dimensional mass scales,

$$\mathcal{L}_N^{\text{Yuk}} \sim L(\phi_S + \phi_U + \phi_I)N^c H\,, \tag{3.8}$$
$$\mathcal{L}_N^{\text{Maj}} \sim N^c(\phi_S + \phi_U + \phi_I)N^c\,, \tag{3.9}$$

where L represents lepton electroweak doublets, and N^c are the CP conjugated right-handed neutrinos. Clearly the S, U subgroups of S_4 are preserved both in the neutrino Yukawa sector and the Majorana sector. Then, after the see-saw mechanism, the same symmetries S, U must be apparent in the effective neutrino mass matrix. Typically both the lepton doublets L and the right-handed neutrinos N^c are taken to be triplets of S_4 or A_4.

The A_4 models in [10] provide a convenient example of the application of A_4 to producing the neutrino flavour symmetry in a direct way. These models are very well known so here we only recall some of the salient features of these models. The group A_4 [3] is a group that describes even permutations of four objects. It has two generators, S and T, and four inequivalent irreducible representations, $\mathbf{1}$, $\mathbf{1}'$, $\mathbf{1}''$ and $\mathbf{3}$. The S generator of the A_4 family symmetry survives in the Majorana sector and becomes part of the neutrino flavour symmetry, while the U transformation of S_4 appears as an accidental symmetry of the neutrino mass matrix arising from the absence of flavons in the $\mathbf{1}', \mathbf{1}''$ representations of A_4. Such A_4 examples satisfy Form Dominance as discussed in [11], i.e. the neutrino mass matrix may be diagonalised by the TB matrix independently of the precise values of the underlying parameters. However in such models each physical neutrino mass results from the VEVs of different flavons, so some mild tuning of flavon VEVs is required in order to achieve a neutrino mass hierarchy, as discussed in [11].

3.3 The flavons of indirect models

In indirect models the family symmetry G_f need not be identified with S_4 or a group containing S_4 as a subgroup. The idea is that the generators S, U, T corresponding to the flavour symmetries respected by $\mathcal{L}^{\text{Maj}}$ and $\mathcal{L}^{\text{Yuk}}$, below the G_f symmetry breaking scale, appear as accidental symmetries. Of course, the family symmetry group G_f will contain symmetry generators, but all elements obtained from these generators will be broken completely by the flavon VEVs. Nevertheless, the combination of flavon VEVs appearing in ϕ^{Maj} will respect the S, U symmetry of the neutrino mass matrix, while the combination of flavon VEVs appearing in ϕ^{Yuk} will respect the T symmetry of the charged lepton mass matrix (at least approximately), even though neither $Z_2^S \times Z_2^U$ nor Z_3^T are subgroups of G_f. In the cases where G_f contains some of the elements of S_4 what happens is that all these elements will be broken by the flavon VEVs, and new ones, analogous to the original ones but in a different basis, will be accidentally restored in the low energy theory.

In indirect models one introduces triplet (or anti-triplet) flavons of the family symmetry G_f, which we refer to as ϕ_1, ϕ_2, ϕ_3, which are arranged to get VEVs v_i in the directions of the orthonormal column vectors in eq. (2.3), namely,

$$\langle\phi_1\rangle = v_1\Phi_1, \quad \langle\phi_2\rangle = v_2\Phi_2, \quad \langle\phi_3\rangle = v_3\Phi_3. \tag{3.10}$$

The flavon VEV alignment ϕ_3 was first discussed in [20] and the analysis was extended to include the flavon VEV alignment ϕ_2 in [4, 5]. The alignment of the flavons ϕ_2, ϕ_3 was subsequently discussed in the framework of discrete family symmetries using F-terms in [7] and using D-terms in [8, 9]. The introduction of the flavon ϕ_1 was discussed in [11]. The possibility that the three flavons ϕ_i could be re-interpreted in terms of string instantons was discussed in [21], although no instanton alignment mechanism was proposed.

In the effective theory the flavons $\phi_{1,2,3}$ are arranged to enter the effective lepton operator quadratically, so that ϕ^{Maj} consists of quadratic combinations of flavons combined as outer products $\phi_1\phi_1^T$, $\phi_2\phi_2^T$ and $\phi_3\phi_3^T$ so that the TB form of the neutrino mass matrix in eq. (2.2) is reproduced when the flavons get their VEVs,

$$\mathcal{L}^{\mathrm{Maj}} \sim \psi \left(\phi_1\phi_1^T + \phi_2\phi_2^T + \phi_3\phi_3^T\right) \psi H H, \tag{3.11}$$

where $\psi \sim \mathbf{3}$ represent lepton doublets in the real (complex) triplet representation of G_f, while ϕ_i are triplets (anti-triplets) of G_f, and we have suppressed flavour indices, order unity coefficients and dimensional mass scales. The required outer product structure of the quadratic flavons generally arises from a see-saw mechanism, as discussed in section 5.

In indirect models the flavon VEVs in eq. (3.10) break all the elements of the underlying family symmetry G_f, while the above quadratic combinations of VEVs preserve all the group elements G of the effective neutrino flavour symmetry,

$$G \left\langle \phi_i\phi_i^T \right\rangle G^T = \left\langle \phi_i\phi_i^T \right\rangle, \tag{3.12}$$

for all $i = 1, 2, 3$, which follows from eqs. (2.9). In fact all the results of the previous section apply here with the column vectors Φ_i replaced by the flavon VEVs $\langle\phi_i\rangle$ in eq. (3.10). In particular from eq. (2.10),

$$G\langle\phi_i\rangle = \eta_i\langle\phi_i\rangle, \tag{3.13}$$

which shows that the flavon VEVs individually break the flavour symmetry of the neutrino mass matrix, corresponding to the group elements in eq. (2.18), due to the presence of the minus signs in the η_i, though their quadratic effect is to preserve the neutrino flavour symmetry. For example, for the group element $S = G_{-+-} = -G_1 + G_2 - G_3$, it is clear from eq. (2.17) that $S\langle\phi_1\rangle = -\langle\phi_1\rangle$, $S\langle\phi_2\rangle = \langle\phi_2\rangle$, $S\langle\phi_3\rangle = -\langle\phi_3\rangle$, that the VEVs $\langle\phi_1\rangle$ and $\langle\phi_3\rangle$ break the symmetry S.

We emphasise that, although the quadratic combinations of flavons preserve an accidental neutrino flavour symmetry, such quadratic combinations will in general break the underlying family symmetry G_f. This does not matter as the only role of G_f is to yield flavon vacuum alignments of the type in eq. (3.10).

4 D-term vacuum alignment in indirect models

The mechanism for vacuum alignment is crucial to the success of any model which purports to explain tri-bimaximal mixing. In the case of direct models, the usual mechanism of vacuum alignment is based on F-term alignment which exploits driving fields in the superpotential as discussed in [10]. This mechanism is also available to indirect models as discussed in [7]. However, in the case of indirect models, an additional and elegant possibility for vacuum alignment becomes available that is not possible for direct models, namely D-term vacuum alignment introduced in [8, 9] as discussed in section 4.1.

The reason that D-term vacuum alignment is not possible in the direct models will be explained below, but is essentially related to the fact that a particular choice of basis is required for the D-term alignment mechanism to work, and this basis is different from that of the neutrino flavour symmetry which must therefore arise accidentally as in indirect models. In fact the D-term vacuum alignment mechanism is so elegant that one may regard it as a primary motivation for considering indirect models rather than direct models.

4.1 Flavon potential and allowed family symmetries G_f

In the previous section we emphasised that the flavons invoked in indirect models generally break the family symmetry G_f completely. An elegant way to obtain the alignments of eq. (2.3) is to start with a flavon scalar potential of the form

$$V \;=\; -\,m^2 \sum_i {\phi^i}^\dagger \phi^i \;+\; \lambda \left(\sum_i {\phi^i}^\dagger \phi^i \right)^2 \;+\; \Delta V\,, \tag{4.1}$$

where the index i labels the components of a particular flavon triplet ϕ and

$$\Delta V \;=\; \kappa \sum_i {\phi^i}^\dagger \phi^i {\phi^i}^\dagger \phi^i\;. \tag{4.2}$$

In a non-supersymmetric theory this potential may simply be written down. However, in a supersymmetric theory, the quartic terms may arise from D-terms, after which this vacuum alignment mechanism is named [8, 9], which take the form

$$\left[\chi^\dagger \chi (\phi^\dagger \phi \phi^\dagger \phi)\right]_D\,,$$

where the F-component of the G_f singlet χ acquires a VEV. Hence supersymmetry is broken and the scalar potential V gets a contribution of the type $\phi^\dagger\phi\phi^\dagger\phi$. The quadratic term on the other hand can originate from a soft supersymmetry breaking mass term, where the mass squared of a given flavon is driven negative by radiative corrections at some scale Λ, leading to a VEV for that flavon set by the scale Λ. As the different flavons have different superpotential couplings to heavy states, and since the soft masses run logarithmically with energy scale, the Λ scales defined above may differ greatly for the different flavons. Thus a hierarchy between the VEVs of various flavon fields is possible, and also stable, in the framework of the radiative breaking mechanism [8, 9].

Only the term ΔV in eqs. (4.1), (4.2) determines the alignment. For $\kappa > 0$ we obtain the alignment Φ_2, while $\kappa < 0$ can give rise to $\widetilde{\Phi}_0 = (1,0,0)^T$. In concrete models, where both of these alignments are typically required, the mixing terms between the two corresponding flavon fields should be suppressed. Although such mixing cannot be forbidden by symmetries, it can be suppressed or even forbidden by invoking messenger arguments. Using SU(3) invariant orthogonality conditions the alignments Φ_3 and Φ_1 can then be obtained successively [8, 22], where additional undesirable invariants may be suppressed or forbidden by a combination of symmetries and messenger arguments. The question we want to pursue in the following is which family symmetries allow for the invariant in eq. (4.2). Obviously, G_f must admit at least one triplet representation. Constraining to special unitary matrices, the generators of the most general symmetry transformations of eq. (4.2) are

$$D = \begin{pmatrix} e^{i\theta_1} & 0 & 0 \\ 0 & e^{i\theta_2} & 0 \\ 0 & 0 & e^{-i(\theta_1+\theta_2)} \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B = -\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.3}$$

Note that these matrices are found among the generators of the C and D series of groups listed long ago by Miller, Blichfeldt and Dickson [23].

Choosing $\theta_1 = 0$ and $\theta_2 = 2\pi l/n$ we immediately recover the group $\Delta(6n^2)$ [24, 25]. Dropping the generator B, we obtain the group $\Delta(3n^2)$ [24, 26] by identifying $\theta_1 = -2\pi(k+l)/n$ and $\theta_2 = 2\pi l/n$. For these two series of non-Abelian finite groups, n, k, l are positive integers with $k, l < n$. A particular choice of l (and k) corresponds to a particular triplet representation of the respective group. The Frobenius group $Z_7 \rtimes Z_3 = T_7$ [13, 27] can be generated from D and A by setting $\theta_1 = \theta_2/2 = 2\pi/7$. Similar groups that have not yet found application as family symmetries, although their order is relatively low, are e.g. $Z_{13} \rtimes Z_3$ with $\theta_1 = \theta_2/3 = 2\pi/13$ or $Z_{19} \rtimes Z_3$ with $\theta_1 = \theta_2/7 = 2\pi/19$, see [28].

4.2 The choice of basis: the $G_f = S_4$ example

It is easy to convince oneself that the neutrino flavour symmetry as given in eq. (2.18) is not a subgroup of any of the above finite groups. In the special case of $\Delta_{24} \cong S_4$, which is generated by

$$S' \equiv D_{(\theta_1=0,\theta_2=\pi)}, \qquad T' \equiv A, \qquad U' \equiv A^2 B, \tag{4.4}$$

one might be tempted to think that, since S_4 is involved, that the D-term alignment mechanism leads to an example of a direct model. However this conclusion would be wrong, since it is clear that the generator S of the neutrino flavour symmetry in eq. (2.13) is not an element of the underlying group defined in the basis of eq. (4.4). In general such arguments, more fully developed below, will make it clear that the D-term alignment mechanism is incompatible with the direct models, but well suited for the indirect models.

We now show that the choice of the basis in eq. (4.3) is crucial for having the invariant in eq. (4.2), and thus for generating the alignment Φ_2. Using a different basis for G_f, the flavon potential ΔV would generally change its form

$$\sum_i {\phi^i}^\dagger \phi^i {\phi^i}^\dagger \phi^i \quad \longrightarrow \quad \sum_{i,j,k,l,m} W_{ji} W^\dagger_{ik} W_{li} W^\dagger_{im} \; {\phi'^j}^\dagger {\phi'}^k {\phi'^l}^\dagger {\phi'}^m, \tag{4.5}$$

where W denotes the unitary basis transformation $\phi' = W\phi$. In this new form, the minima of ΔV are usually much less apparent. However, since we already know how the flavon potential is minimised in the original basis, we can trace back the structure of the vacuum for ϕ' to the alignment of ϕ. It will turn out that no "nice" alignment can be obtained for ϕ'. To be specific, let us consider the case $\kappa > 0$. In the original basis, the flavon potential

ΔV leads to an alignment

$$\widetilde{\Phi}_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i\vartheta_1} \\ e^{i\vartheta_2} \\ e^{i\vartheta_3} \end{pmatrix} . \tag{4.6}$$

It is important to notice that the phases are completely undetermined by the flavon potential of eq. (4.2). However these phases are unphysical since they would correspond to having phases in the second column of U_{TB} which can be absorbed into the charged lepton fields. What matters is the magnitude of the components of eq. (4.6) and the orthogonality of the flavon VEVs $\widetilde{\Phi}_1, \widetilde{\Phi}_2, \widetilde{\Phi}_3$. In indirect models it is a pure convention to set the phases of eq. (4.6) to zero, leading to Φ_2. Nonetheless, the potential ΔV of eq. (4.2) is minimised by the more general alignment of eq. (4.6). Changing to the basis ϕ', an analogous continuum of minima will persist, however, the explicit structure of the set of alignment vectors can appear arbitrary.

For example in $G_f = S_4$, suppose we perform a unitary basis transformation [10]

$$W = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} , \qquad \omega = e^{2\pi i/3} , \tag{4.7}$$

to take us from the basis of eq. (4.4) to the neutrino flavour basis of eq. (2.13),

$$S = W S' W^\dagger , \quad T = W T' W^\dagger , \quad U = W U' W^\dagger = U' . \tag{4.8}$$

Note that the third generator is identical in both bases. Nonetheless the neutrino flavour symmetry $Z_2^S \times Z_2^U$ is not a subgroup of S_4 generated by S', T', U'. Performing the vacuum alignment in the basis of eq. (4.4), would lead to eq. (4.6), which would appear in the neutrino flavour basis as,

$$\widetilde{\Phi}'_2 = W \widetilde{\Phi}_2 = \frac{1}{3} \begin{pmatrix} e^{i\vartheta_1} + e^{i\vartheta_2} + e^{i\vartheta_3} \\ e^{i\vartheta_1} + \omega e^{i\vartheta_2} + \omega^2 e^{i\vartheta_3} \\ e^{i\vartheta_1} + \omega^2 e^{i\vartheta_2} + \omega e^{i\vartheta_3} \end{pmatrix} . \tag{4.9}$$

In general the original alignment would appear very arbitrary in the neutrino flavour basis, as can be seen from the following examples,

$$W \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad W \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} , \quad W \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ i \\ -i \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ 1-\sqrt{3} \\ 1+\sqrt{3} \end{pmatrix} .$$

These examples illustrate how dramatically the alignment vectors $\widetilde{\Phi}'_2$ in the new basis change with the phases ϑ_i. Since all of them minimise the flavon potential in the basis ϕ', the invariant of eq. (4.5) does not seem to give rise to useful alignments. That is not to say that a basis transformation alters physics! Rather from the *practical* point of view the choice of a good basis is important in devising an indirect model, starting from "nice" flavon alignments which are then coupled to the fermions to generate their mass matrices.

To summarise, in the S_4 example with D-term vacuum alignment, the above argument shows that the model should be constructed in the S', T', U' basis of eq. (4.4) where the alignment term in the potential is given by eq. (4.2) leading to the vacuum alignment $\widetilde{\Phi}_2$ in eq. (4.6). With the $G_f = S_4$ model constructed in the basis S', T', U', the resulting neutrino mass matrix would have a flavour symmetry corresponding to the generators S, U. Due to the difference between the two bases the neutrino flavour symmetry $Z_2^S \times Z_2^U$ is not a subgroup of the group generated by S', U', T' even though $U = U'$. Furthermore, working in a convenient convention where the phases may be dropped, i.e. $\widetilde{\Phi}_2 \to \Phi_2$, the flavon alignments Φ_1, Φ_2, Φ_3 guarantee that S_4 is broken completely: Φ_1 breaks all elements of S_4, while Φ_2 and Φ_3 are left invariant by *distinct* generators,

$$T'\Phi_2 \;=\; \Phi_2\,, \qquad U'\Phi_3 \;=\; \Phi_3\;. \tag{4.10}$$

Having at least two flavons contributing to the neutrino sector, the underlying family symmetry will therefore be broken completely. We conclude that in the $G_f = S_4$ family symmetry model with D-term vacuum alignment, the neutrino flavour symmetry arises accidentally, with the flavon alignment Φ_2 being directly enforced by the G_f invariant term in eq. (4.2). The alignment of the other flavons Φ_1, Φ_3 of the indirect model arise from orthogonality arguments and are not enforced directly from a G_f invariant like eq. (4.2), making their alignment more model dependent, see e.g. [22]. This has important phenomenological implications, as discussed later in the context of an indirect $G_f = A_4$ example where similar comments apply.

4.3 Additional invariants non-grata

Working in the basis of eq. (4.3), we have seen that various underlying non-Abelian discrete family symmetries are conceivable. They all allow for the two quartic invariants of type $\mathbf{\overline{3}3\overline{3}3}$, where all triplets are assumed to be identical,

$$\mathcal{I}_0 \;=\; \left(\sum_i \phi^{i\dagger}\phi^i\right)^2, \qquad \mathcal{I}_1 \;=\; \sum_i \phi^{i\dagger}\phi^i\phi^{i\dagger}\phi^i\;. \tag{4.11}$$

Depending on G_f there may however be additional invariants which we identify in the following.

- $Z_7 \rtimes Z_3$: The symmetric Kronecker product of two triplets reads [13, 27]

  $$(\mathbf{3}\times\mathbf{3})_s \;=\; \mathbf{3} + \mathbf{\overline{3}}\;.$$

  Multiplying this with its complex conjugate, i.e.

  $$(\mathbf{3}\times\mathbf{3})_s \times \left(\mathbf{\overline{3}}\times\mathbf{\overline{3}}\right)_s \;=\; \left(\mathbf{3}+\mathbf{\overline{3}}\right)\times\left(\mathbf{\overline{3}}+\mathbf{3}\right)\,,$$

  it is evident that only two independent invariants of type $\mathbf{\overline{3}3\overline{3}3}$ are possible, namely $\mathcal{I}_0$ and $\mathcal{I}_1$. The result is identical for the groups $Z_{13} \rtimes Z_3$ and $Z_{19} \rtimes Z_3$ where $(\mathbf{3}\times\mathbf{3})_s = \mathbf{3}' + \mathbf{\overline{3}}$ with $\mathbf{3}'$ different from $\mathbf{\overline{3}}$.

- $\Delta(3n^2)$: In this case the triplets are labelled by two indices k, l. The symmetric product of two identical $\mathbf{3}_{(k,l)}$ becomes [26]

$$\left(\mathbf{3}_{(k,l)} \times \mathbf{3}_{(k,l)}\right)_s \;=\; \mathbf{3}_{(2k,2l)} + \mathbf{3}_{(-k,-l)} \,.$$

Assuming that $\mathbf{3}_{(k,l)}$ is a faithful representation of $\Delta(3n^2)$, the two symmetric triplets on the right-hand side are different triplets for $n > 3$. Therefore, we find only two invariants in these cases, $\mathcal{I}_0$ and $\mathcal{I}_1$.

With $n = 2$ we recover $\Delta_{12} \cong A_4$. As there is only one real triplet in A_4, the first symmetric triplet decomposes into a sum of one-dimensional representations,

$$n = 2: \quad (\mathbf{3} \times \mathbf{3})_s \;=\; \mathbf{1} + \mathbf{1}' + \mathbf{1}'' + \mathbf{3}\,,$$

giving rise to four independent invariants. In addition to those of eq. (4.11) we get [9]

$$\mathcal{I}_2 \;=\; \phi^1\phi^1\phi^{2\dagger}\phi^{2\dagger} + \phi^2\phi^2\phi^{3\dagger}\phi^{3\dagger} + \phi^3\phi^3\phi^{1\dagger}\phi^{1\dagger}\,, \tag{4.12}$$

and its complex conjugate $\mathcal{I}_{\bar{2}} = \mathcal{I}_2^\dagger$.

With $n = 3$ we obtain the group Δ_{27} with one triplet and its complex conjugate. In this case, both symmetric triplets are identical,

$$n = 3: \quad (\mathbf{3} \times \mathbf{3})_s \;=\; \overline{\mathbf{3}} + \overline{\mathbf{3}}\,,$$

yielding four independent invariants. Besides $\mathcal{I}_0$ and $\mathcal{I}_1$ which have been applied in [8] we find (compare [13])

$$\mathcal{I}_3 \;=\; \phi^1\phi^1\phi^{2\dagger}\phi^{3\dagger} + \phi^2\phi^2\phi^{3\dagger}\phi^{1\dagger} + \phi^3\phi^3\phi^{1\dagger}\phi^{2\dagger}\,, \tag{4.13}$$

and its complex conjugate $\mathcal{I}_{\bar{3}} = \mathcal{I}_3^\dagger$.

- $\Delta(6n^2)$: For these groups there are two types of triplets ($q = 1, 2$) which differ by a sign for the generator B; each type of triplet is labelled by an additional index l. The symmetric Kronecker products take the form [25]

$$\left(\mathbf{3}_{\mathbf{q}(l)} \times \mathbf{3}_{\mathbf{q}(l)}\right)_s \;=\; \mathbf{3}_{\mathbf{1}(2l)} + \mathbf{3}_{\mathbf{1}(-l)} \,.$$

With $\mathbf{3}_{\mathbf{q}(l)}$ being a faithful representation of $\Delta(6n^2)$, the two triplets on the right-hand side are different triplets for $n > 3$, so that only two quartic invariants are obtained, $\mathcal{I}_0$ and $\mathcal{I}_1$.

With $n = 2$ we have $\Delta_{24} \cong S_4$, which has only two real triplets. Furthermore, the first symmetric triplet decomposes into a one-dimensional plus a two-dimensional representation,

$$n = 2: \quad (\mathbf{3}_\mathbf{q} \times \mathbf{3}_\mathbf{q})_s \;=\; \mathbf{1} + \mathbf{2} + \mathbf{3}_\mathbf{1}\,,$$

from which we can construct three independent invariants. In addition to those of eq. (4.11) we find

$$\mathcal{I}_4 \;=\; \mathcal{I}_2 + \mathcal{I}_{\bar{2}}\,. \tag{4.14}$$

With $n = 3$ the group Δ_{54} is generated, which has two complex triplets and their conjugates. The symmetric Kronecker products are

$$\left(\mathbf{3}_{\mathbf{q}(l)} \times \mathbf{3}_{\mathbf{q}(l)}\right)_s = \mathbf{3}_{\mathbf{1}(-l)} + \mathbf{3}_{\mathbf{1}(-l)} \,, \tag{4.15}$$

leading to four independent invariants: $\mathcal{I}_0$, $\mathcal{I}_1$, $\mathcal{I}_3$ and $\mathcal{I}_{\bar{3}}$. Note that these are identical to the invariants of Δ_{27}.

As we are interested in family symmetries which give rise to "nice" flavon alignments, it seems desirable to pick G_f such that only the invariants in eq. (4.11) are allowed. Additional invariants may spoil the structure of the vacuum derived from ΔV of eq. (4.2) unless they are sufficiently suppressed. From that point of view, $\Delta\left(3n^2\right)$ and $\Delta\left(6n^2\right)$ with $n > 3$, as well as $Z_7 \rtimes Z_3$ together with some other examples such as $Z_{13} \rtimes Z_3$ and $Z_{19} \rtimes Z_3$ are preferred candidates for the underlying family symmetry in indirect models.

4.4 Alternative useful invariants

We conclude this section by mentioning that one could alternatively make use of other invariants in the flavon potential that give rise to vacuum alignments in indirect models. One such example could be the term

$$\mathcal{I}_5 = \phi^1\phi^2\phi^3 \,, \tag{4.16}$$

which is left invariant under D and A of eq. (4.3). In other words, the underlying discrete family symmetry G_f in such a model could be $Z_7 \rtimes Z_3$ or $\Delta(3n^2)$. One easily finds that the flavon potential with

$$\Delta V = \kappa\left(\mathcal{I}_5 + \mathcal{I}_5^\dagger\right) \,, \qquad \kappa < 0 \,,$$

is minimised by an alignment of type $\widetilde{\Phi}_2$,

$$\frac{1}{\sqrt{3}}\begin{pmatrix} e^{i\vartheta_1} \\ e^{i\vartheta_2} \\ e^{-i(\vartheta_1+\vartheta_2)} \end{pmatrix} .$$

5 The see-saw mechanism in indirect models

The type I see-saw mechanism [19] in indirect models exploits the flavons ϕ_i in eq. (3.10) which are either triplets or anti-triplets of G_f depending on whether the representations are complex. There are two approaches to the see-saw mechanism in indirect models, depending on whether the right-handed neutrinos N^c are singlets or triplets under the family symmetry G_f, while the left-handed leptons L are always triplets of G_f. In both cases we shall show how the quadratic combinations of flavons preserve an accidental neutrino flavour symmetry of the neutrino mass matrix, in the effective Lagrangian after the see-saw mechanism has taken place.

5.1 Right-handed neutrino singlets $N_i^c \sim 1$

Consider the see-saw Lagrangian,

$$\mathcal{L}_N^{\mathrm{Yuk}} \sim L_i \left(\phi_1^i N_1^c + \phi_2^i N_2^c + \phi_3^i N_3^c\right) H \,, \tag{5.1}$$

$$\mathcal{L}_N^{\mathrm{Maj}} \sim M_1 N_1^c N_1^c + M_2 N_2^c N_2^c + M_3 N_3^c N_3^c \,, \tag{5.2}$$

where the diagonal forms of eqs. (5.1), (5.2) require additional symmetries. Note that the Yukawa Lagrangian only involves the flavons ϕ_i linearly, not quadratically, and therefore does not respect the flavour symmetry of the neutrino mass matrix which only emerges after the see-saw mechanism. Since $N_i^c \sim \mathbf{1}$, the combination of $L_i \sim \mathbf{3}$ and $\phi_i \sim \mathbf{3}$ (or $\phi_i \sim \overline{\mathbf{3}}$ if the representations are complex) must yield a singlet of G_f. It is important to note that these models are formulated in a basis where the family indices are trivially summed over. After the see-saw mechanism takes place, this results in an effective Lagrangian of the form of eq. (3.11),

$$\mathcal{L}^{\mathrm{Maj}} \sim L \left(\frac{\phi_1 \phi_1^T}{M_1} + \frac{\phi_2 \phi_2^T}{M_2} + \frac{\phi_3 \phi_3^T}{M_3} \right) LHH. \tag{5.3}$$

Thus we see the appearance of the quadratic combinations of flavons which serve to preserve an accidental neutrino flavour symmetry of the neutrino mass matrix, in the effective Lagrangian after the see-saw mechanism has taken place. In matrix notation, when the flavons get their VEVs in eq. (3.10) the three columns of the Dirac mass matrix M_D are proportional to the VEVs of the three flavons, and the right-handed neutrino mass matrix is diagonal,

$$M_D = (a_1 \Phi_1, a_2 \Phi_2, a_3 \Phi_3), \quad M_{RR} = \mathrm{diag}(M_1, M_2, M_3), \tag{5.4}$$

where a_i are constants. The resulting effective neutrino mass matrix is thus

$$M^\nu = M_D M_{RR}^{-1} M_D^T = \frac{a_1^2}{M_1} \Phi_1 \Phi_1^T + \frac{a_2^2}{M_2} \Phi_2 \Phi_2^T + \frac{a_3^2}{M_3} \Phi_3 \Phi_3^T \,, \tag{5.5}$$

which is of TB form in eq. (2.2) where we identify the physical neutrino mass eigenvalues as $m_1 = a_1^2/M_1$, $m_2 = a_2^2/M_2$, $m_3 = a_3^2/M_3$. This also corresponds to Form Dominance [11], with each column of M_D constructed from the VEV of a different flavon, so no flavon VEV tuning is required to achieve a neutrino mass hierarchy. This corresponds to so-called Natural Form Dominance [11].

The A_4 model in [9] provides a convenient example of the general mechanism described above for the realisation of neutrino flavour symmetry in an indirect way using right-handed neutrino singlets. It also provides an example of the use of CSD [4, 18] to generate a strong neutrino mass hierarchy. The lepton doublets are taken to be triplets of A_4, $L \sim \mathbf{3}$, while the right-handed neutrinos and right-handed charged leptons are all taken to be trivial singlets of A_4. In this model the see-saw Lagrangian is taken to be, at leading order, similar to the form in eqs. (5.1), (5.2), but with some of the right-handed neutrinos re-labelled, and involves the A_4 triplet flavons $\phi_2^i \sim \mathbf{3}$, $\phi_3^i \sim \mathbf{3}$ plus a new triplet flavon $\phi_0^i \sim \mathbf{3}$,

$$\mathcal{L}_N^{\mathrm{Yuk}} \sim L_i \left(\phi_3^i N_1^c + \phi_2^i N_2^c + \phi_0^i N_3^c\right) H \,, \tag{5.6}$$

$$\mathcal{L}_N^{\mathrm{Maj}} \sim M_1 N_1^c N_1^c + M_2 N_2^c N_2^c + M_3 N_3^c N_3^c \,, \tag{5.7}$$

where the assumed forms in eqs. (5.6), (5.7) require additional symmetries as discussed in [9]. The flavons ϕ_2, ϕ_3 have the alignments as in eq. (3.10) while the flavon ϕ_0 has an alignment of the form,

$$\langle\phi_0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_0 \equiv \Phi_0 v_0 \ . \tag{5.8}$$

The charged lepton Yukawa Lagrangian also takes a similar form to eq. (5.6), with the flavon ϕ_0 being responsible for the third family charged lepton Yukawa coupling,

$$\mathcal{L}^{\rm Yuk}_{\rm lep} \sim L_i(\phi_3^i e^c_R + \phi_2^i \mu^c_R + \phi_0^i \tau^c_R)H \ . \tag{5.9}$$

The resulting charged lepton mass matrix is approximately diagonal due to the strong charged lepton mass hierarchy with the dominant $(3,3)$ Yukawa coupling provided by the ϕ_0 VEV. As pointed out in section 4.1, the required hierarchy between different flavon VEVs can be obtained in the framework of radiative symmetry breaking. The actual model in [9] is more complicated than this, since it unifies the quarks and leptons into a Pati-Salam gauge group, under which for example the right-handed neutrinos are not singlets, and shows how all the fermion mass hierarchies and mass splittings may be achieved.

In this A_4 model the neutrino flavour symmetry arises accidentally as a result of two effects. The first effect is CSD [4, 18] in which the right-handed neutrino mass M_3 is so heavy[2] that the third right-handed neutrino N_3^c effectively decouples from the see-saw mechanism, rendering the effect of the flavon ϕ_0 on the see-saw mechanism irrelevant [18]. The second effect is that the remaining two flavons ϕ_2, ϕ_3 which are relevant for the see-saw mechanism lead to an effective Majorana Lagrangian after the see-saw mechanism of the form of eq. (5.3),

$$\mathcal{L}^{\rm Maj} \sim L\left(\frac{\phi_3\phi_3^T}{M_1} + \frac{\phi_2\phi_2^T}{M_2}\right)LHH. \tag{5.10}$$

Thus we see the appearance of the quadratic combinations of flavons which serve to preserve an accidental neutrino flavour symmetry of the neutrino mass matrix, in the effective Lagrangian after the see-saw mechanism has taken place. However, since the charged lepton mass matrix is only approximately diagonal, TB neutrino mixing will receive corrections from the charged lepton mixing angles [29].

An important feature of the indirect A_4 model is that it is formulated in an $SO(3)$ type basis for which the product of two triplets $a = (a_1, a_2, a_3) \sim \mathbf{3}$ and $b = (b_1, b_2, b_3) \sim \mathbf{3}$ contains the invariant singlet given by the diagonal combination $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \sim \mathbf{1}$. In the A_4 example [9] this implies that the generators in the triplet representation are given by S', T' of eq. (4.4), see also [3]. On the other hand these models reproduce the neutrino flavour symmetry $Z_2^S \times Z_2^U$, which however, is not a subgroup of the original A_4 family symmetry. The reason for this is twofold: (i) the generator U is not an element of A_4, and (ii) the bases are different. Furthermore, the VEV of ϕ_3 breaks all elements generated

[2]In the concrete model [9], additional symmetries are adopted so that the third term on the r.h.s. of eq. (5.7) arises at a lower-dimensional level compared to the first and second term.

from S', T', while ϕ_2 preserves the subgroup $Z_3^{T'}$, the net effect being that no subgroup of the underlying A_4 symmetry can survive in either the neutrino or the charged lepton sector.

Contrary to the ϕ_2 alignment which is obtained directly from the invariant in eq. (4.2), the ϕ_3 alignment emerges from subsequent orthogonality arguments and is thus more model dependent, see e.g. [22]. While the alignment of ϕ_2 may be altered by subleading correction, it is possible for the VEV of ϕ_3 to receive a *leading order* alignment of the form $(\epsilon, 1, -1)$, where the value of ϵ is potentially sizable [22]. This would then result in a reactor angle of order $\theta_{13} \sim \epsilon/\sqrt{2}$, at the same time maintaining accurately the tri-bimaximal solar and atmospheric predictions [22]. Such tri-bimaximal-reactor mixing of the type discussed in [22] would then be a prediction of indirect models with such a leading order ϕ_3 alignment.

5.2 Right-handed neutrino triplets $N_i^c \sim 3$

In this case the see-saw Lagrangian is taken to have the form, at leading order,

$$\mathcal{L}_N^{\text{Yuk}} \sim L_i \left(\phi_1^i\phi_1^j + \phi_2^i\phi_2^j + \phi_3^i\phi_3^j\right) N_j^c H\,, \tag{5.11}$$

$$\mathcal{L}_N^{\text{Maj}} \sim N_i^c \left(\phi_1^i\phi_1^j + \phi_2^i\phi_2^j + \phi_3^i\phi_3^j\right) N_j^c\,, \tag{5.12}$$

where the Yukawa and Majorana Lagrangians are required to be diagonal in the flavon types, up to a relabelling of right-handed neutrinos. Note that in this case both the Yukawa and Majorana Lagrangian involve the flavons ϕ_i quadratically, and therefore in this case the high energy see-saw theory respects the same flavour symmetry of the neutrino mass matrix which will emerge after the see-saw mechanism, although of course the family symmetry G_f is not respected by these quadratic flavon combinations.

As before we see the appearance of the quadratic combinations of flavons which serve to preserve an accidental neutrino flavour symmetry of the neutrino mass matrix, in the effective Lagrangian after the see-saw mechanism has taken place. In matrix notation, when the flavons get their VEVs in eq. (3.10), the Dirac mass matrix M_D is proportional to a sum over outer products of the VEVs of the three flavons, while the right-handed neutrino mass matrix takes the form of the TB neutrino mass matrix in eq. (2.2),

$$M_D = a_1\Phi_1\Phi_1^T + a_2\Phi_2\Phi_2^T + a_3\Phi_3\Phi_3^T\,, \tag{5.13}$$

$$M_{RR} = M_1\Phi_1\Phi_1^T + M_2\Phi_2\Phi_2^T + M_3\Phi_3\Phi_3^T\,, \tag{5.14}$$

where a_i are constants and M_i are the right-handed neutrino mass eigenvalues. Clearly the right-handed neutrino mass matrix is diagonalised by the tri-bimaximal mixing matrix,

$$U_{TB}^T M_{RR} U_{TB} = \text{diag}\,(M_1,\ M_2,\ M_3)\,. \tag{5.15}$$

Hence we can write

$$M_{RR}^{-1} = \frac{\Phi_1\Phi_1^T}{M_1} + \frac{\Phi_2\Phi_2^T}{M_2} + \frac{\Phi_3\Phi_3^T}{M_3}\,, \tag{5.16}$$

using eq. (2.4). The resulting effective neutrino mass matrix is thus, from eqs. (5.13), (5.16), using the orthonormality relations in eq. (2.5),

$$M^\nu = M_D M_{RR}^{-1} M_D^T = \frac{a_1^2}{M_1}\Phi_1\Phi_1^T + \frac{a_2^2}{M_2}\Phi_2\Phi_2^T + \frac{a_3^2}{M_3}\Phi_3\Phi_3^T, \tag{5.17}$$

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as in eq. (2.2), where we identify the physical neutrino mass eigenvalues as $m_1 = a_1^2/M_1$, $m_2 = a_2^2/M_2$, $m_3 = a_3^2/M_3$.

The $G_f = \Delta_{27}$ model in [8] provides an example of the general mechanism described above for the realisation of neutrino flavour symmetry in an indirect way using right-handed neutrino triplets. It also provides an example of the use of CSD [4, 18] to generate a strong neutrino mass hierarchy $m_1 \ll m_{2,3}$ due to a very heavy third right-handed neutrino of mass M_3. The Yukawa Lagrangian is of the leading order form,

$$\mathcal{L}_N^{\mathrm{Yuk}} \sim L_i \left(\phi_3^i \phi_2^j + \phi_2^i \phi_3^j + \phi_0^i \phi_0^j \right) N_j^c H \,, \tag{5.18}$$

leading to a Dirac mass matrix of the form,

$$M_D = a_1 \Phi_3 \Phi_2^T + a_2 \Phi_2 \Phi_3^T + a_0 \Phi_0 \Phi_0^T , \tag{5.19}$$

where a_i are constants. The Majorana Lagrangian is of the leading order form,

$$\mathcal{L}_N^{\mathrm{Maj}} \sim N_i^c \left(\phi_2^i \phi_2^j + \phi_3^i \phi_3^j + \phi_0^i \phi_0^j \right) N_j^c \,, \tag{5.20}$$

leading to the heavy Majorana masses,

$$M_{RR} = M_1 \Phi_2 \Phi_2^T + M_2 \Phi_3 \Phi_3^T + M_3 \Phi_0 \Phi_0^T \,, \tag{5.21}$$

where M_i are the right-handed neutrino mass eigenvalues where we assume $M_1 < M_2 \ll M_3$. Assuming that the heaviest right-handed neutrino of mass M_3 approximately decouples, according to the CSD mechanism, the resulting effective neutrino mass matrix is thus, from eqs. (5.19), (5.21), using the orthonormality relations in eq. (2.5),

$$M^\nu = M_D M_{RR}^{-1} M_D^T \approx \frac{a_1^2}{M_1} \Phi_3 \Phi_3^T + \frac{a_2^2}{M_2} \Phi_2 \Phi_2^T , \tag{5.22}$$

similar to eq. (2.2), where $m_1 \ll m_{2,3}$ is negligible and the remaining physical neutrino mass eigenvalues are $m_3 = a_1^2/M_1$, $m_2 = a_2^2/M_2$. Note that in this case the Yukawa Lagrangian involves off-diagonal flavon bilinears $\phi_i \phi_j$ (i.e. two different flavon types) and therefore it does not respect the flavour symmetry of the neutrino mass matrix which only emerges after the see-saw mechanism.

6 Conclusion

In this paper we have shown that the flavour symmetry of the neutrino mass matrix may originate from an underlying family symmetry in two quite distinct ways, either directly or indirectly.

The direct models are typically based on S_4 family symmetry or any family symmetry that contains S_4 as a subgroup (such as for example $PSL(2,7) = \Sigma(168)$ [15]) whose generators S, U are directly preserved in the Majorana sector sector. In the case of direct models the neutrino flavour symmetry appears directly as a result of a $Z_2^S \times Z_2^U$ subgroup of S_4 being preserved in the effective Majorana Lagrangian by the flavons ϕ_S, ϕ_U, but not ϕ_T, appearing in the Majorana sector. In the see-saw implementation of the direct models the neutrino flavour symmetry is preserved also by the high energy see-saw theory, due to the presence of the flavons ϕ_S, ϕ_U, but not ϕ_T, appearing in the see-saw Dirac and Majorana Lagrangians as in eqs. (3.8), (3.9).

The main focus of the paper has been on indirect models which are based on any family symmetry G_f which is completely broken and in which the observed lepton flavour symmetry emerges as an accidental symmetry. In the case of indirect models the only role of the underlying family symmetry G_f is to yield flavon alignments with ϕ_i proportional to the three columns of U_{TB} as in eq. (3.10). The flavon VEVs of indirect models break the underlying family symmetry G_f, so that no remnant of this symmetry survives in the low energy theory. The origin of the neutrino flavour symmetry is completely accidental due to the quadratic appearance of such flavons in the effective Majorana Lagrangian.

Comparing the direct models to the indirect models it is seen that, for the direct models, the flavons ϕ_S, ϕ_U appear in the Majorana sector as in eq. (3.7), while for the indirect models it is the quadratic flavon combinations $\phi_i \phi_i^T$ which appear in the Majorana sector as in eq. (3.11), where such quadratic combinations accidentally preserve the neutrino flavour symmetry and reproduce the TB neutrino mass matrix.

We have emphasised that the necessary vacuum alignments of indirect models can be achieved using an elegant D-term vacuum alignment mechanism, together with orthogonality arguments, and we have catalogued the possible choices of family symmetry G_f which are consistent with this mechanism. In this way we are led to the large classes of possible candidate family symmetries: $\Delta(3n^2)$ and $\Delta(6n^2)$, as well as $Z_7 \rtimes Z_3 = T_7$ together with similar examples such as $Z_{13} \rtimes Z_3$ and $Z_{19} \rtimes Z_3$ which so far have been ignored as potentially interesting candidates for G_f. Although the presence of the underlying family symmetry G_f is crucial for producing such D-term vacuum alignments, we have shown that it does not include the neutrino flavour symmetry $Z_2^S \times Z_2^U$ as a subgroup which must therefore emerge as an accidental symmetry. We have explicitly shown how this works for the case of $G_f = S_4$ where D-term vacuum alignment implies an indirect, rather than direct, realisation of the neutrino flavour symmetry.

We have seen that the see-saw implementation of the indirect models depends on whether the right-handed neutrinos are singlets or triplets of the family symmetry G_f and may be summarised as follows:

(i) If the right-handed neutrinos are singlets, then the flavons ϕ_i appear linearly in the Dirac Lagrangian, as in eq. (5.1), so that the high energy see-saw theory does not respect the low energy neutrino flavour symmetry of the resulting low energy effective theory in eq. (5.3) which only involves the quadratic flavon combinations.

(ii) If the right-handed neutrinos are triplets, then the quadratic flavon combinations $\phi_i \phi_i^T$ may appear in both the high energy Dirac and Majorana Lagrangians, as in eqs. (5.11), (5.12), which means that the high energy see-saw theory will respect the low energy neutrino flavour symmetry in such cases. However in other cases of indirect models in which the right-handed neutrinos are triplets, the right-handed neutrinos may be ordered differently so that an off-diagonal flavon combination $\phi_i \phi_j^T$

may appear in the high energy Dirac Lagrangian, as in eq. (5.18), with the diagonal combinations $\phi_i \phi_i^T$ only emerging in the low energy effective Majorana Lagrangian so that the neutrino flavour symmetry only arises below the see-saw scale.

We have seen that explicit realistic see-saw implementations of indirect models are typically based on CSD which implies a strong neutrino hierarchy and involves only the two flavons with alignments along the second and third columns of the TB mixing matrix. We have sketched existing examples of such models based on $\Delta_{12} = A_4$ and Δ_{27}.

In conclusion, the main point we want to make in this paper is that the family symmetry group G_f need not have anything directly to do with the observed neutrino flavour symmetry. In such indirect models the only role of G_f is to produce the vacuum alignments proportional to the columns of U_{TB}, as in eq. (3.10), where the quadratic appearance of such flavons in the effective Majorana Lagrangian is responsible for the observed neutrino flavour symmetry corresponding to TB mixing. In such indirect models, D-term vacuum alignment provides an elegant vacuum alignment mechanism, not available to the direct models. In fact the D-term vacuum alignment mechanism is so elegant that one may regard it as a primary motivation for considering indirect models rather than direct models. The D-term vacuum alignment mechanism only requires G_f to contain triplet representations of the form of eq. (4.3) which is the case for the large classes of finite groups mentioned above. Typically the VEV of the flavon ϕ_2, proportional to the second column of U_{TB}, will be derived directly from the G_f invariant term in eq. (4.2), whereas the alignments of the other flavons such as ϕ_3, proportional to the third column of U_{TB}, are subsequently obtained using orthogonality arguments. In such indirect models the alignment of ϕ_3 is therefore generally more model dependent and, for example, could lead to a large reactor angle while preserving the TB solar and atmospheric angle predictions [22], providing a smoking gun signature of the indirect origin of the neutrino flavour symmetry.

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